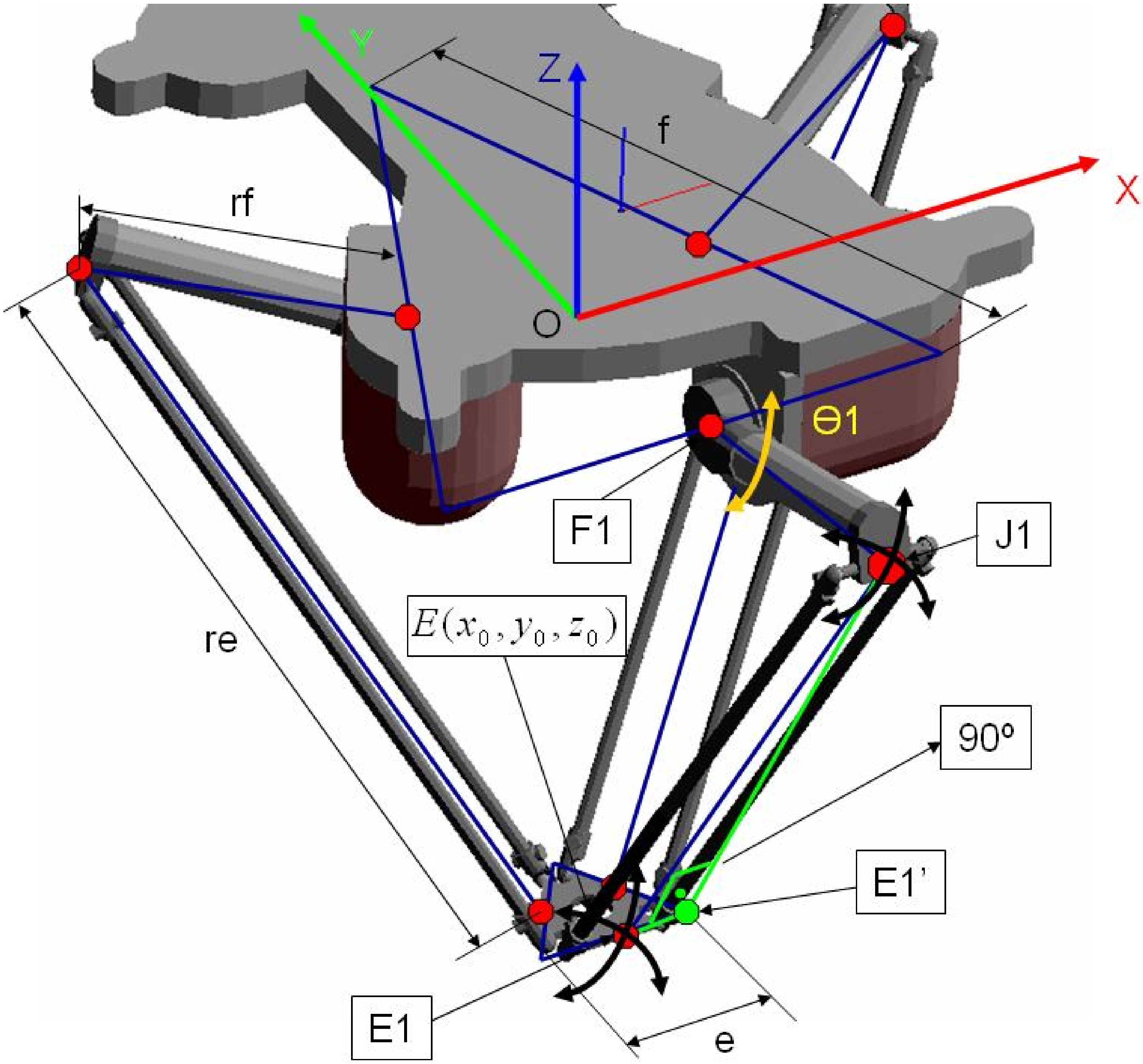
Inverse Kinematics of a Delta Robot.

We begin by extracting a kinematic model in which the parameters are as follows:

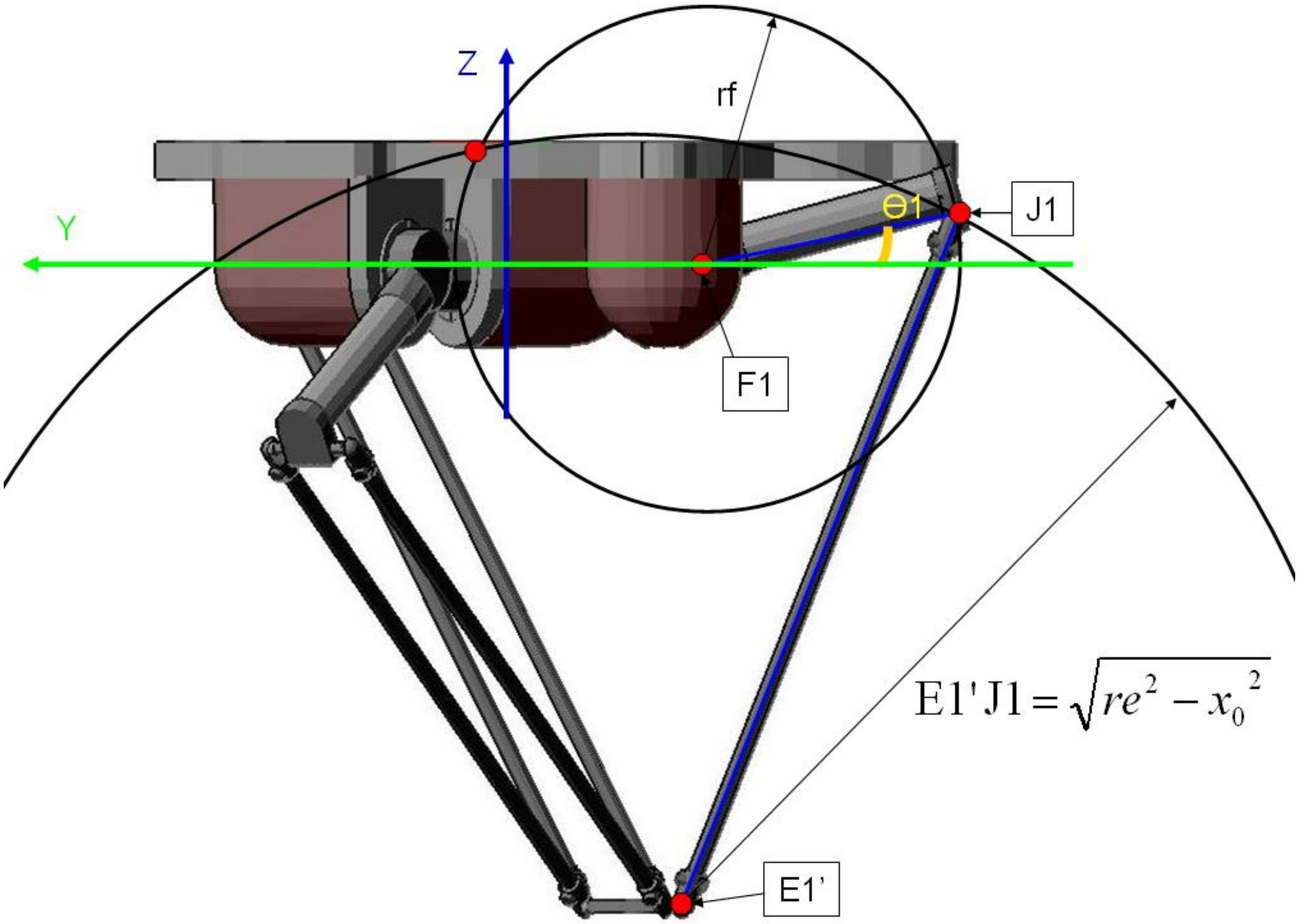
- Length of one side of the equilateral triangle that forms the framework: f
- Length of one side of the equilateral triangle that forms the effector: e
- Length from the fixed joint F1 until the spherical joint J1: rf
- Length from the spherical joint J1 until the spherical joint E1: re

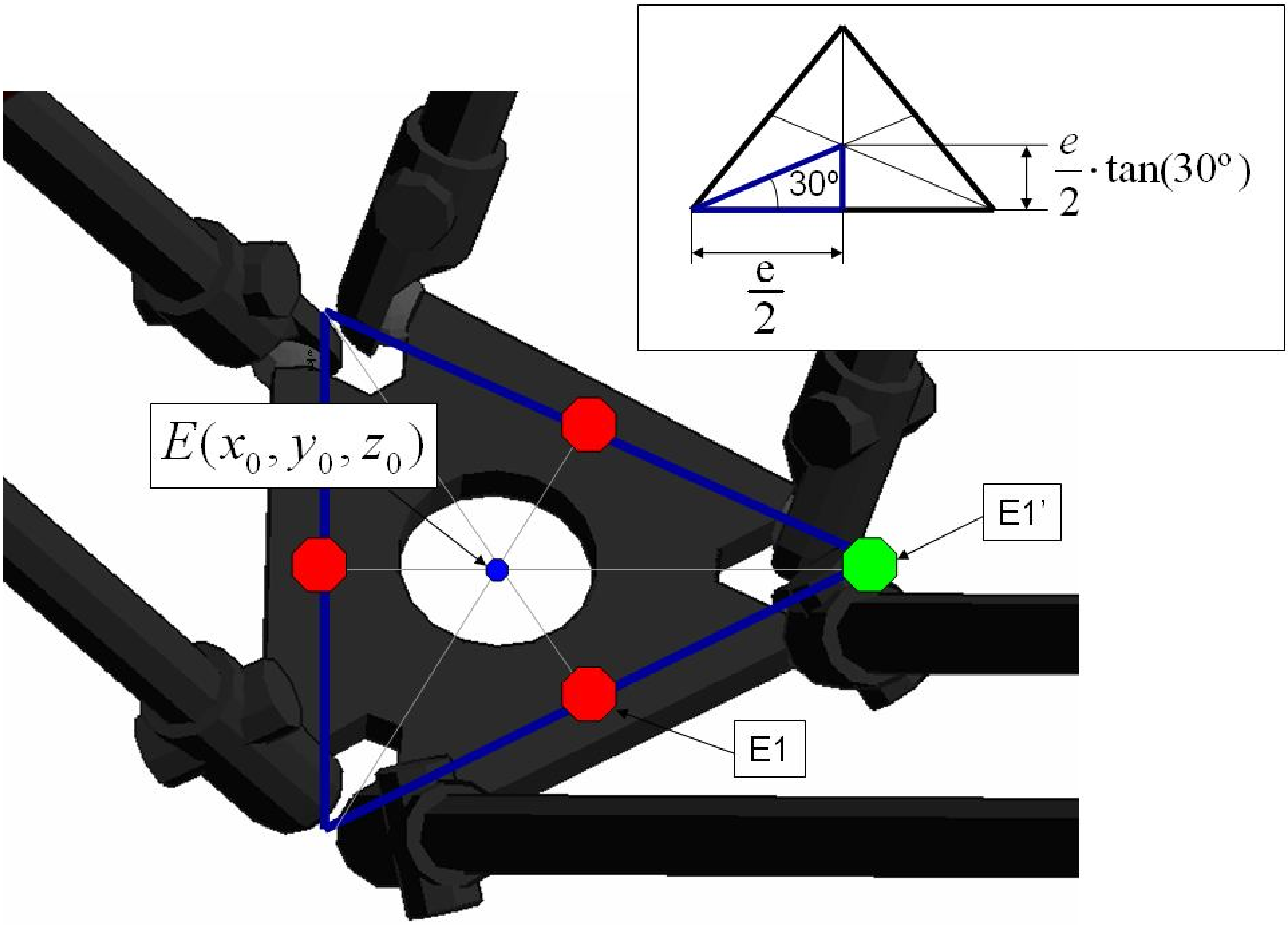


The point J1 can be found as the intersection of two circles. One with center in F1 and radius rf .

Other with center in E1' and radius: $\sqrt{re^2 - x_0{}^2}$ as we calculate later.

We should choose only one intersection, the smaller Y coordinate solution.





$E = (x_0, y_0, z_0)$

$EE1 = \frac{e}{2}\tan(30^o) = \frac{e}{2\sqrt{3}}$

$$\left.\begin{aligned} E1 &= \left(x_0, y_0 - \frac{e}{2\sqrt{3}}, z_0\right) \\ E1' &= \left(0, y_0 - \frac{e}{2\sqrt{3}}, z_0\right) \end{aligned}\right\} E1E1' = x_0$$

$F1 = \left(0, \frac{f}{2\sqrt{3}}, 0\right)$

$E1'J1 = \sqrt{E1J1^2 - E1E1'^2} = \sqrt{re^2 - {x_0}^2} \rightarrow radius\ of\ the\ circle$

We can get the coordinates of J1 in terms of the coordinates of E and the parameters of the machine

$$\begin{cases} (y_{J1} - y_{F1})^2 + (z_{J1} - z_{F1})^2 = rf^2 \\ (y_{J1} - y_{E1'})^2 + (z_{J1} - z_{E1'})^2 = re^2 - x_0^2 \end{cases}$$

$$\begin{cases} \left(y_{J1} - \frac{f}{2\sqrt{3}}\right)^2 + (z_{J1})^2 = rf^2 \\ \left(y_{J1} - y_0 + \frac{e}{2\sqrt{3}}\right)^2 + (z_{J1} - z_0)^2 = re^2 - x_0^2 \end{cases}$$

$$J1(0, y_{J1}, z_{J1}) = f(rf, re, f, e, x_0, y_0, z_0)$$

Inverse kinematics solution:

$$\theta_1 = a\tan\left(\frac{z_{J1}}{y_{F1} - y_{J1}}\right)$$

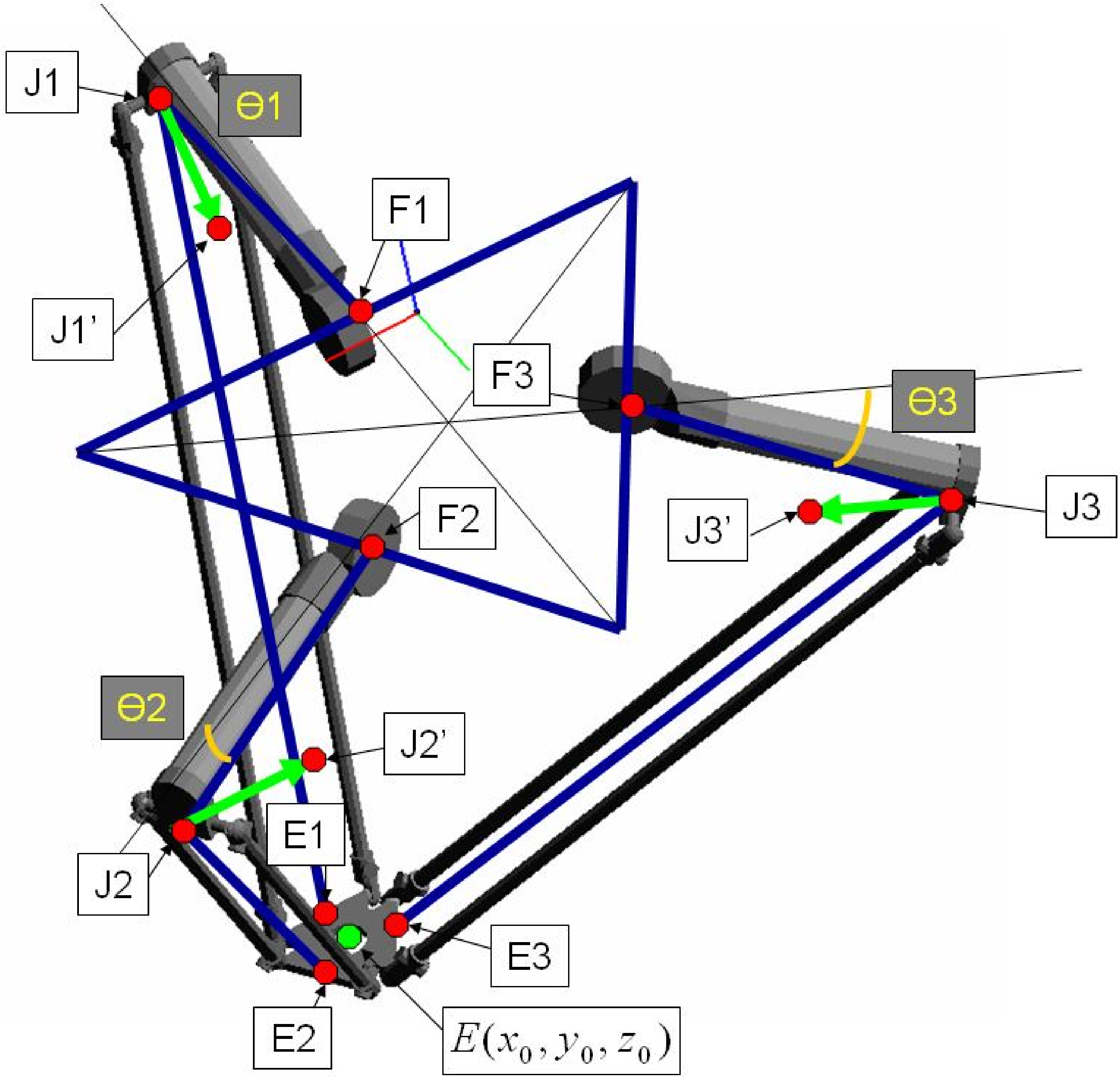
$$\theta_2 \ y \ \theta_3 \rightarrow E_0(x_0, y_0, z_0) \rightarrow E_0'(x_0', y_0', z_0')$$

$$\begin{cases} x_0' = x \cdot \cos(\pm 120^o) + y \cdot \sin(\pm 120^o) \\ y_0' = -x \cdot \sin(\pm 120^o) + y \cdot \cos(\pm 120^o) \\ z_0' = z_0 \end{cases}$$

Forward Kinematics of a Delta Robot.

To calculate the direct kinematics we move the center of the spheres from points J1, J2, J3 to the points J1', J2', J3' using the transition vectors $\overline{E1E0}$, $\overline{E2E0}$, $\overline{E3E0}$ respectively.

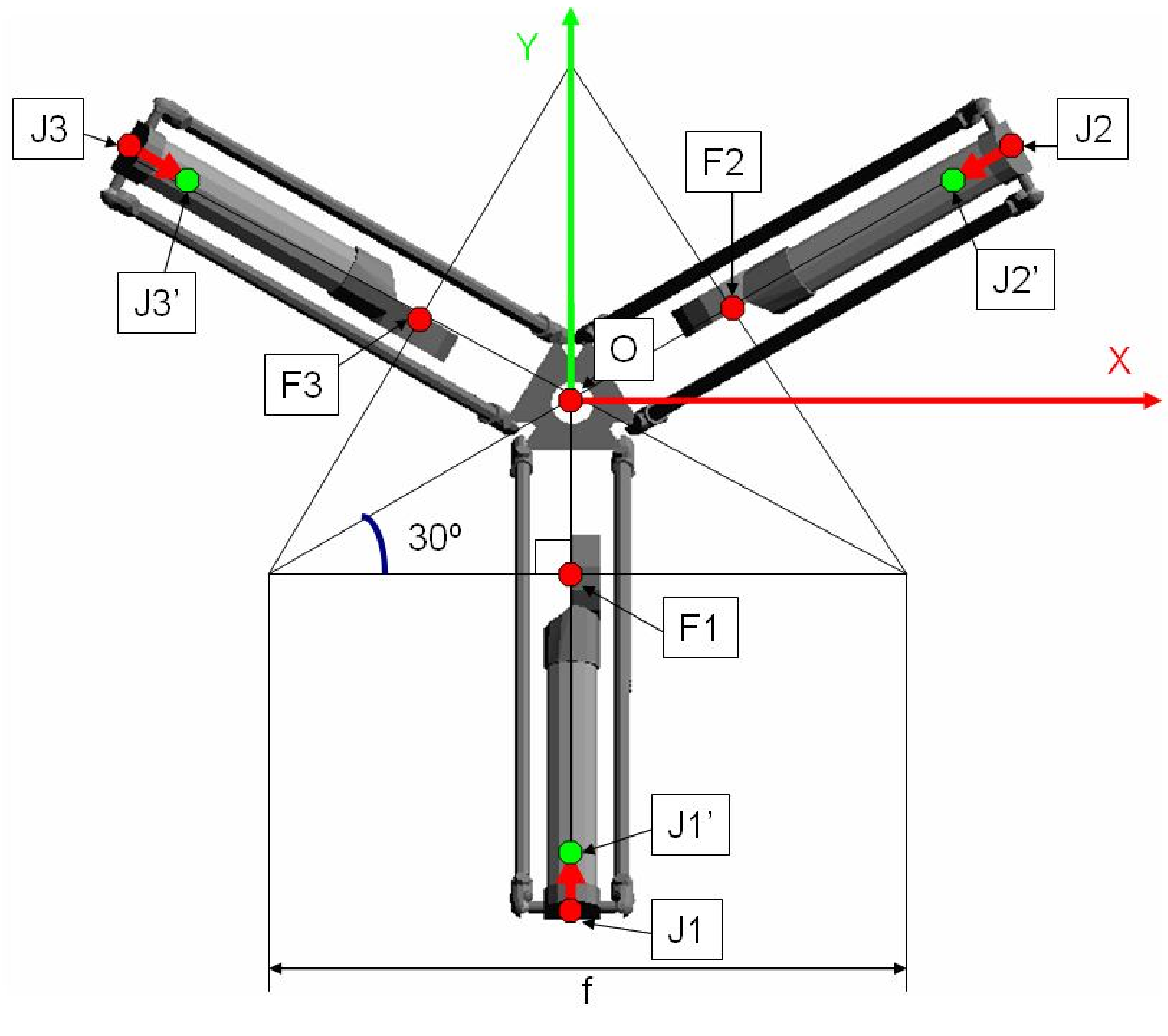
After this transition the three spheres will intersect in the point: $E(x_0, y_0, z_0)$



To find coordinates $E(x_0, y_0, z_0)$ of point E=E0 we need to solve three equations like:

$$(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 = re^2 \rightarrow j = 1, 2, 3$$

and (x, y, z) the coordinates of sphere centers J1', J2', J3'.



$$OF1 = OF2 = OF3 = \frac{f}{2}\tan(30^{\circ}) = \frac{f}{2\sqrt{3}}$$

$$J1J1' = J2J2' = J3J3' = \frac{e}{2}\tan(30^{\circ}) = \frac{e}{2\sqrt{3}}$$

$$F1J1 = rf \cdot \cos(\theta_1)$$
$$F1J2 = rf \cdot \cos(\theta_2)$$
$$F3J3 = rf \cdot \cos(\theta_3)$$

$$J1' = \left(0 \quad \frac{-(f-e)}{2\sqrt{3}} - rf \cdot \cos(\theta_1) \quad -rf \cdot \sin(\theta_1)\right) = (x_1, y_1, z_1)$$

$$J2' = \left(\left[\frac{f-e}{2\sqrt{3}} + rf \cdot \cos(\theta_2)\right]\cos(30^o) \quad \left[\frac{f-e}{2\sqrt{3}} + rf \cdot \cos(\theta_2)\right]\sin(30^o) \quad -rf \cdot \sin(\theta_2)\right) = (x_2, y_2, z_2)$$

$$J3' = \left(\left[\frac{f-e}{2\sqrt{3}} + rf \cdot \cos(\theta_3)\right]\cos(30^o) \quad \left[\frac{f-e}{2\sqrt{3}} + rf \cdot \cos(\theta_3)\right]\sin(30^o) \quad -rf \cdot \sin(\theta_3)\right) = (x_3, y_3, z_3)$$

$$\begin{cases} (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = re^2 \\ (x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2 = re^2 \\ (x-x_3)^2 + (y-y_3)^2 + (z-z_3)^2 = re^2 \end{cases}$$

$$\begin{cases} x^2 + y^2 + z^2 - 2y_1 y - 2z_1 z = re^2 - y_1^2 - z_1^2 \quad [1] \\ x^2 + y^2 + z^2 - 2x_2 x - 2y_2 y - 2z_2 z = re^2 - x_2^2 - y_2^2 - z_2^2 \quad [2] \\ x^2 + y^2 + z^2 - 2x_3 x - 2y_3 y - 2z_3 z = re^2 - x_3^2 - y_3^2 - z_3^2 \quad [3] \end{cases}$$

$$w_i = x_i^2 + y_i^2 + z_i^2$$

$$\begin{cases} [4] = [1] - [2] \\ [5] = [1] - [3] \\ [6] = [2] - [3] \end{cases} \rightarrow \begin{cases} x_2 x + (y_1 - y_2) y + (z_1 - z_2) z = (w_1 - w_2)/2 \\ x_3 x + (y_1 - y_3) y + (z_1 - z_3) z = (w_1 - w_3)/2 \\ (x_2 - x_3) x + (y_2 - y_3) y + (z_2 - z_3) z = (w_2 - w_3)/2 \end{cases}$$

From [4] and [5]

$$[7] \quad x = a_1 z + b_1$$

$$a_1 = \frac{1}{d}\left[(z_2 - z_1)(y_3 - y_1) - (z_3 - z_2)(y_2 - y_1)\right]$$

$$b_1 = -\frac{1}{2d}\left[(w_2 - w_1)(y_3 - y_1) - (w_3 - w_1)(y_2 - y_1)\right]$$

$$d = (y_2 - y_1)x_3 - (y_3 - y_1)x_2$$

$$[8] \quad y = a_2 z + b_2$$

$$a_2 = \frac{-1}{d}\left[(z_2 - z_1)(x_3) - (z_3 - z_1)(x_2)\right]$$

$$b_2 = \frac{1}{2d}\left[(w_2 - w_1)(x_3) - (w_3 - w_1)(x_2)\right]$$

Substitute [7] and [8] in [1]

$$(a_1^2 + a_2^2 + 1)z^2 + 2(a_1 + a_2(b_2 - y) - z_1)z + (b_1^2 + (b_2 - y_1)^2 + z_1^2 - re^2)$$

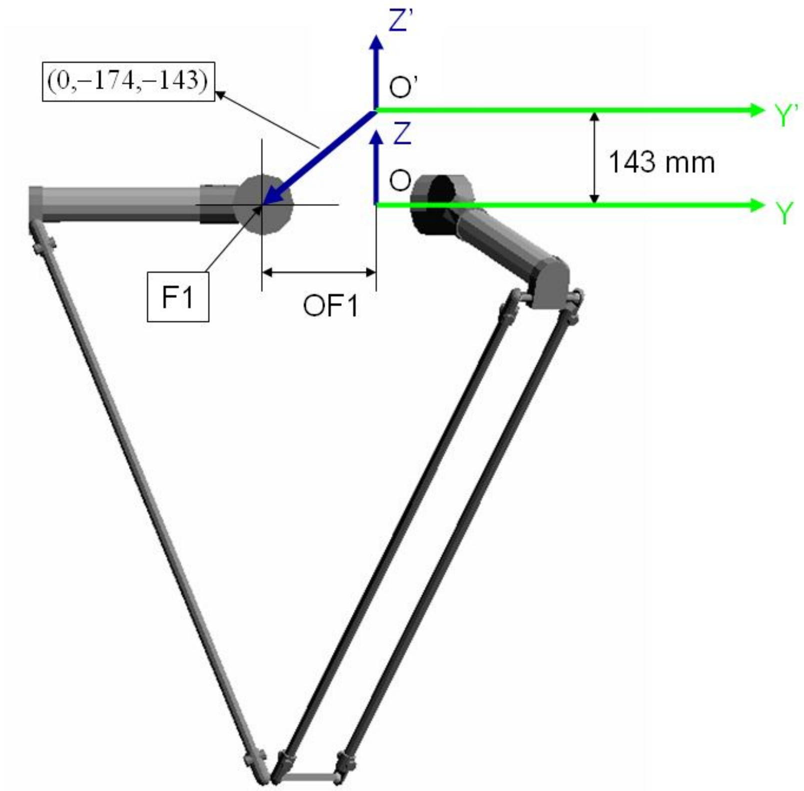
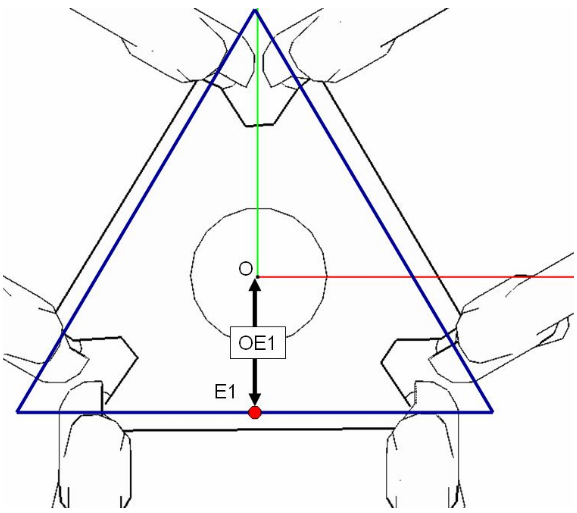
The smallest negative equation root results $z = z_0$

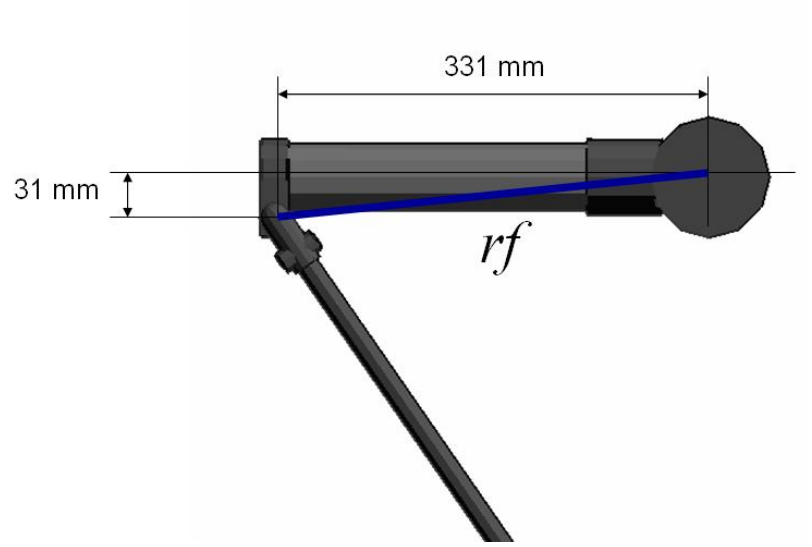
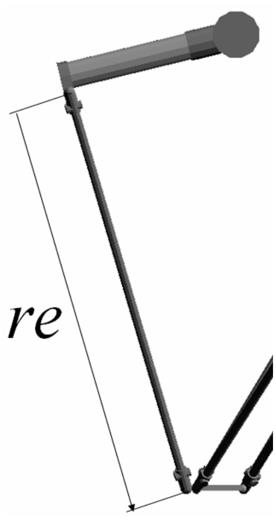
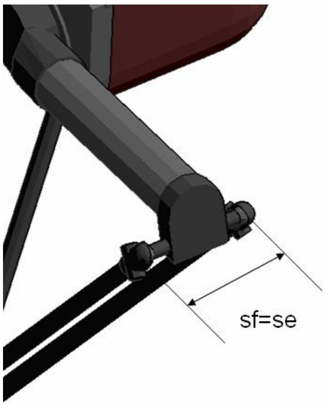
x_0, y_0 are calculate with [7] and [8] equations

Veltru D12 Robot Parameters

Note: The parameter sf=se not interfere with the calculations of direct and inverse kinematics.

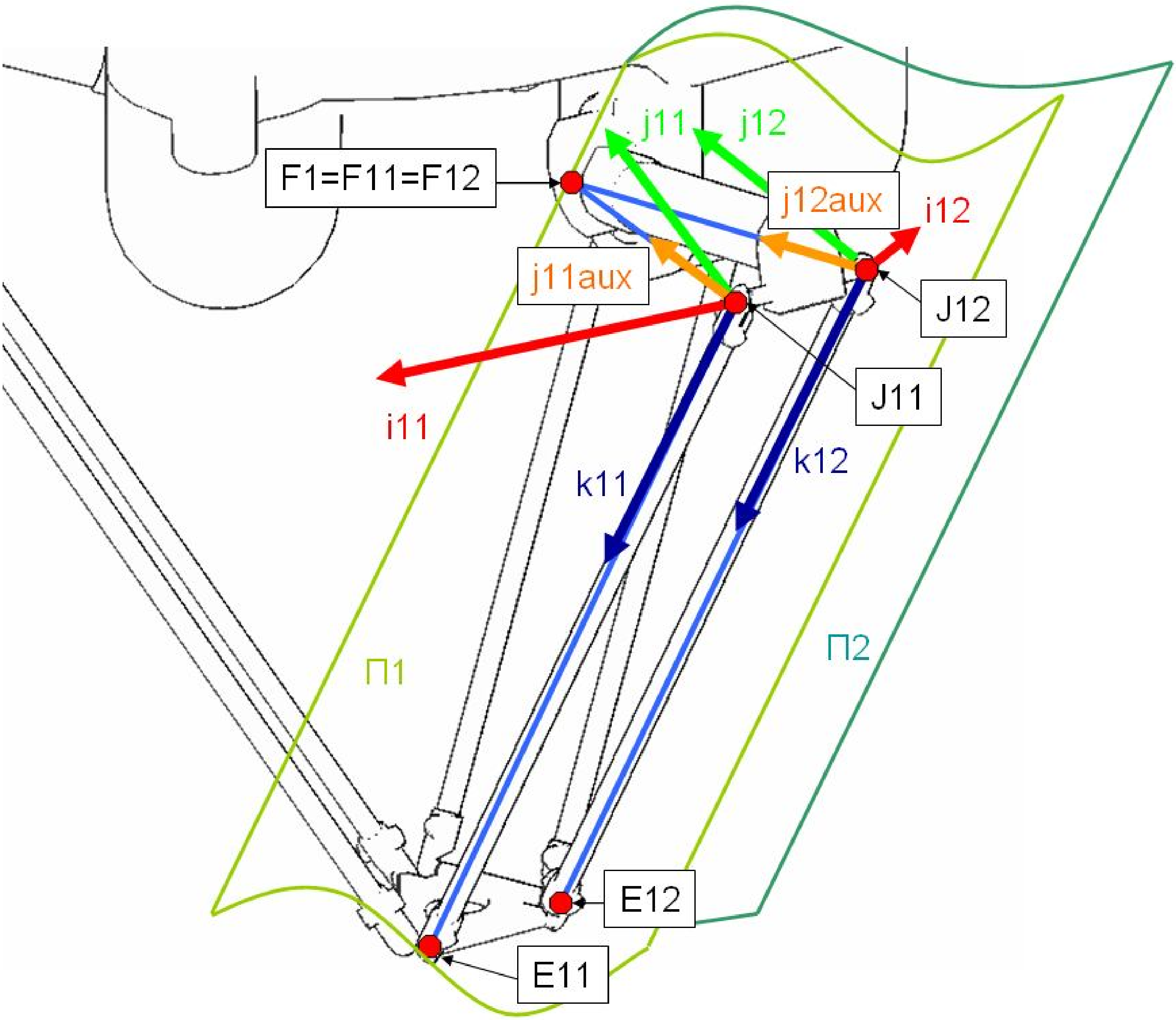
It is used to draw pairs of arms correctly, this is to calculate its orientation and position.

f	$4 \cdot OF1 \cdot \cos(30^{o}) = 4 \cdot 174 \cdot \cos(30^{o}) = 602.75mm$ 
e	$4 \cdot OE1 \cdot \cos(30^{o}) = 4 \cdot 43 \cdot \cos(30^{o}) = 148.95mm$ 

rf	$\sqrt{331^2 + 31^2} = 332.44mm$ 
re	$870mm$ 
$sf = se$	$50mm$ 

Spherical Joint Orientations

When representing the 3D kinematic model, we find it's necessary to calculate the orientation of the pairs of connecting rods ball joints. The following shows how to calculate the orientation.



The length between J11 and J12 is the parameter sf=se. This allows us to calculate F11 F12 from F1 and E11 E12 from E1.

Knowing the point J11 and E11, we can calculate the unit vector of the line that connecting two points. This is k11 vector.

Knowing the point J11 and F11, we can calculate the unit vector of the line that connecting two points. This is j11aux vector.

With this two vectors (k11, j11aux) we can calculate Π1 plane.

The general equation of the plane Π1 : $Ax + By + Cz + D = 0$ defines the principal vector i11 (A, B, C)

With the two orthonormal vectors (i11, k11) we can calculate the third orthonormal vector j11 and form the system MCB that defines the position-orientation of the spheric joint J11 (Tx, Ty, Tz, Rx, Ry, Rz)

$$MCB = \begin{bmatrix} ix & jx & kx & Tx \\ iy & jy & ky & Ty \\ iz & jz & kz & Tz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$if \quad ix = 0 \rightarrow ix = 10^{-5} \rightarrow Rz = a\tan\left(-\frac{jx}{ix}\right)$$

$$if \quad kz = 0 \rightarrow kz = 10^{-5} \rightarrow Rx = a\tan\left(-\frac{ky}{kz}\right)$$

$$Ry = a\tan\left(\frac{\cos(Rx) \cdot kx}{kz}\right)$$

Repeating these steps for the second joint we got the system that defines the position-orientation of the spheric joint J12